

## ON OCCULT PERIOD MAPS

STEPHEN KUDLA  
AND  
MICHAEL RAPOPORT

*In memoriam Jonathan Rogawski*

## Abstract

We interpret the “occult” period maps of Allcock/Carlson/Toledo [2, 3], resp. of Looijenga/Swierstra [20, 21], resp. of Kondo [13, 14] in moduli theoretic terms, as a construction of certain families of polarized abelian varieties of Picard type. We show that these period maps are morphisms defined over their natural field of definition.

## 1. INTRODUCTION

In papers of Allcock/Carlson/Toledo [2, 3], resp. of Looijenga/Swierstra [20, 21], resp. of Kondo [13, 14], “hidden” period maps are constructed in certain cases. The target spaces of these maps are certain *arithmetic quotients of complex unit balls*. The basic observation, which is the starting point of this paper, is that these arithmetic quotients can be interpreted as the complex points of certain *moduli spaces of abelian varieties of Picard type*, of the kind considered in our paper [19]. Consequently, the purpose in this paper is to interpret these hidden period maps in moduli-theoretic terms. The pay-off of this exercise is that we can raise and partially answer some *descent problems* which seem natural from our view point, and which are related to a similar descent problem addressed by Deligne in [7] in his theory of *complete intersections of Hodge level one*.

Why do we speak of “hidden”, or “occult” period maps in this context? This is done in order to make the distinction with the usual period maps which associate to a family of smooth projective complex varieties (over some base scheme  $S$ ) the (polarized) Hodge structures of its fibers, which then induces a map from  $S$  to a quotient by a discrete group of a period domain. Let us recall three examples of classical period maps:

- (1) **Case of quartic surfaces.** In this case the period map is a holomorphic map of orbifolds

$$\varphi : \text{Quartics}_{2,\mathbb{C}}^{\circ} \rightarrow V(2, 19)/\Gamma.$$

Here  $\text{Quartics}_{2,\mathbb{C}}^{\circ}$  denotes the stack parametrizing smooth quartic surfaces up to projective equivalence,

$$\text{Quartics}_{2,\mathbb{C}}^{\circ} = [\mathbb{P}\text{Sym}^3(\mathbb{C}^4)^{\circ}/\text{PGL}_4]$$

(stack quotient in the orbifold sense). The target space is the orbifold quotient of the space of oriented positive 2-planes in a quadratic space  $V$  of signature  $(2, 19)$  by the automorphism group  $\Gamma$  of a lattice in  $V$ .

- (2) **Case of cubic threefolds.** In this case the period map is a holomorphic map of orbifolds

$$\varphi : \text{Cubics}_{3,\mathbb{C}}^{\circ} \rightarrow \mathfrak{H}_5/\Gamma.$$

Here  $\text{Cubics}_{3,\mathbb{C}}^\circ$  denotes the stack parametrizing smooth cubic threefolds up to projective equivalence. The target space is the orbifold quotient of the Siegel upper half space of genus 5 by the Siegel group  $\Gamma = \text{Sp}_5(\mathbb{Z})$ .

(3) **Case of cubic fourfolds.** In this case the period map is a holomorphic map of orbifolds

$$\varphi : \text{Cubics}_{4,\mathbb{C}}^\circ \rightarrow V(2, 20)/\Gamma.$$

Here  $\text{Cubics}_{4,\mathbb{C}}^\circ$  denotes the stack parametrizing smooth cubic fourfolds up to projective equivalence. The target space is the orbifold quotient of the space of oriented positive 2-planes in a quadratic space  $V$  of signature  $(2, 20)$  by the automorphism group  $\Gamma$  of a lattice in  $V$ .

In the first case, by the Torelli theorem of Piatetskii-Shapiro/Shafarevich, the map  $\varphi$  is an open embedding. In the second case, by the Torelli theorem of Clemens/Griffiths, the map  $\varphi$  is a locally closed embedding (it is not an open embedding since the source of  $\varphi$  has dimension 10, and the target has dimension 15). In the third case, by the Torelli theorem of Voisin, the map  $\varphi$  is an open embedding.

The construction of the occult period maps is quite different, although it does use the classical period maps indirectly. For instance, the construction of Allcock/Carlson/Toledo attaches a certain Hodge structure to any smooth cubic surface which allows one to distinguish between non-isomorphic ones, even though the natural Hodge structures on the cohomology in the middle dimension of all cubic surfaces are isomorphic. Also, in one dimension higher, their construction allows them to define an *open embedding* of the stack of cubic threefolds into an arithmetic quotient of the complex unit ball of dimension 10.

Our second aim in this paper is to identify the complements of the images of occult period maps with *special divisors* considered in [19].

The lay-out of the paper is as follows. In sections 2, 3 and 4 we recall some of the theory and notation of [19]. In sections 5, 6, 7, and 8, respectively, we explain in turn the case of cubic surfaces, cubic threefolds, curves of genus 3, and curves of genus 4. In section 9, we explain the descent problem, and solve it in zero characteristic. In the final section, we make a few supplementary remarks.

We stress that the proofs of our statements are all contained in the papers mentioned above, and that our work only consists in interpreting these results.

We thank B. van Geemen, D. Huybrechts and E. Looijenga for very helpful discussions. We also thank J. Achter for keeping us informed about his progress in proving our conjecture in section 9 in some cases.

## 2. MODULI SPACES OF PICARD TYPE

Let  $\mathbf{k} = \mathbb{Q}(\sqrt{\Delta})$  be an imaginary-quadratic field with discriminant  $\Delta$ , ring of integers  $O_{\mathbf{k}}$ , and a fixed complex embedding. We write  $a \mapsto a^\sigma$  for the non-trivial automorphism of  $O_{\mathbf{k}}$ .

For integers  $n \geq 1$  and  $r$ ,  $0 \leq r \leq n$ , we consider the groupoid  $\mathcal{M} = \mathcal{M}(n-r, r) = \mathcal{M}(\mathbf{k}; n-r, r)$  fibered over  $(\text{Sch}/O_{\mathbf{k}})$  which associates to an  $O_{\mathbf{k}}$ -scheme  $S$  the groupoid of triples  $(A, \iota, \lambda)$ . Here  $A$  is an abelian scheme over  $S$ ,  $\lambda$  is a principal polarization, and  $\iota : O_{\mathbf{k}} \rightarrow \text{End}(A)$  is a homomorphism such that

$$\iota(a)^* = \iota(a^\sigma),$$

for the Rosati involution  $*$  corresponding to  $\lambda$ . In addition, the following signature condition is imposed

$$\text{char}(T, \iota(a) \mid \text{Lie } A) = (T - i(a))^{n-r} \cdot (T - i(a^\sigma))^r, \quad \forall a \in O_{\mathbf{k}}, \quad (2.1)$$

where  $i : O_{\mathbf{k}} \rightarrow \mathcal{O}_S$  is the structure map.

We will mostly consider the complex fiber  $\mathcal{M}_{\mathbb{C}} = \mathcal{M} \times_{\text{Spec } O_{\mathbf{k}}} \text{Spec } \mathbb{C}$  of  $\mathcal{M}$ . In any case,  $\mathcal{M}$  is a Deligne-Mumford stack and  $\mathcal{M}_{\mathbb{C}}$  is smooth.

We will also have to consider the following variant, defined by modifying the requirement above that the polarization  $\lambda$  be principal. Let  $d > 1$  be a square free divisor of  $|\Delta|$ . Then  $\mathcal{M}(\mathbf{k}; d; n - r, r)^* = \mathcal{M}(\mathbf{k}; n - r, r)^*$  parametrizes triples  $(A, \iota, \lambda)$  as in the case of  $\mathcal{M}(\mathbf{k}; n - r, r)$ , except that we impose the following condition on  $\lambda$ . We require first of all that  $\ker \lambda \subset A[d]$ , so that  $O_{\mathbf{k}}/(d)$  acts on  $\ker \lambda$ . In addition, we require that this action factor through the quotient ring  $\prod_{p \mid d} \mathbb{F}_p$  of  $O_{\mathbf{k}}/(d)$ , and that  $\lambda$  be of degree  $d^{n-1}$ , if  $n$  is odd, resp.  $d^{n-2}$ , if  $n$  is even. In the notation introduced in section 13 of [19], we have  $\mathcal{M}(\mathbf{k}; d; n - r, r)^* = \mathcal{M}(\mathbf{k}; \mathbf{t}; n - r, r)^{*, \text{naive}}$ , where the function  $\mathbf{t}$  on the set of primes  $p$  with  $p \mid \Delta$  assigns to  $p$  the integer  $2[(n-1)/2]$  if  $p \mid d$ , and 0 if  $p \nmid d$ . Note that if  $\mathbf{k}$  is the Gaussian field  $\mathbf{k} = \mathbb{Q}(\sqrt{-1})$ , then necessarily  $d = 2$ ; if  $\mathbf{k}$  is the Eisenstein field  $\mathbf{k} = \mathbb{Q}(\sqrt{-3})$ , then  $d = 3$ .

### 3. COMPLEX UNIFORMIZATION

Let us recall from [19] the complex uniformization of  $\mathcal{M}(\mathbf{k}; n - 1, 1)(\mathbb{C})$  in the special case that  $\mathbf{k}$  has class number one. For  $n > 2$ , let  $(V, (\cdot, \cdot))$  be a hermitian vector space over  $\mathbf{k}$  of signature  $(n - 1, 1)$  which contains a self-dual  $O_{\mathbf{k}}$ -lattice  $L$ . By the class number hypothesis,  $V$  is unique up to isomorphism. When  $n$  is odd, or when  $n$  is even and  $\Delta$  is odd, the lattice  $L$  is also unique up to isomorphism. We assume that one of these conditions is satisfied. Let  $\mathcal{D}$  be the space of negative lines in the  $\mathbb{C}$ -vector space  $(V_{\mathbb{R}}, \mathbb{I}_0)$ , where the complex structure  $\mathbb{I}_0$  is defined in terms of the discriminant of  $\mathbf{k}$ , as  $\mathbb{I}_0 = \sqrt{\Delta}/|\sqrt{\Delta}|$ . Let  $\Gamma$  be the isometry group of  $L$ . Then the complex uniformization is the isomorphism of orbifolds,

$$\mathcal{M}(\mathbf{k}; n - 1, 1)(\mathbb{C}) \simeq [\Gamma \backslash \mathcal{D}].$$

There is an obvious  $*$ -variant of this uniformization, which gives

$$\mathcal{M}(\mathbf{k}; n - 1, 1)^*(\mathbb{C}) \simeq [\Gamma^* \backslash \mathcal{D}],$$

where  $\Gamma^*$  is the automorphism group of the (*parahoric*) lattice  $L^*$  corresponding to the  $*$ -moduli problem. The lattice  $L^*$  is uniquely determined up to isomorphism by the condition that there is a chain of inclusions of  $O_{\mathbf{k}}$ -lattices  $L^* \subset (L^*)^\vee \subset (\sqrt{d})^{-1}L^*$ , with quotient  $(L^*)^\vee/L^*$  of dimension  $n - 1$  if  $n$  is odd and  $n - 2$  if  $n$  is even, when localized at any prime ideal  $\mathfrak{p}$  dividing  $d$ . Here, for an  $O_{\mathbf{k}}$ -lattice  $M$  in  $V$ , we write

$$M^\vee = \{ x \in V \mid h(x, L) \subset O_{\mathbf{k}} \}$$

for the dual lattice.

### 4. SPECIAL CYCLES (KM-CYCLES)

We continue to assume that the class number of  $\mathbf{k}$  is one, and recall from [19] the definition of special cycles over  $\mathbb{C}$ . Let  $(E, \iota_0)$  be an elliptic curve with  $CM$  by  $O_{\mathbf{k}}$  over  $\mathbb{C}$ , which we fix in what follows. Note that, due to our class number hypothesis,  $(E, \iota_0)$  is unique up to

isomorphism. We denote its canonical principal polarization by  $\lambda_0$ . For any  $\mathbb{C}$ -scheme  $S$ , and  $(A, \iota, \lambda) \in \mathcal{M}(\mathbf{k}; n-1, 1)(S)$ , let

$$V'(A, E) = \mathrm{Hom}_{O_{\mathbf{k}}}(E_S, A) ,$$

where  $E_S = E \times_{\mathbb{C}} S$  is the constant elliptic scheme over  $S$  defined by  $E$ . Then  $V'(A, E)$  is a projective  $O_{\mathbf{k}}$ -module of finite rank with a positive definite  $O_{\mathbf{k}}$ -valued hermitian form given by

$$h'(x, y) = \lambda_0^{-1} \circ y^{\vee} \circ \lambda \circ x \in \mathrm{End}_{O_{\mathbf{k}}}(E_S) = O_{\mathbf{k}} .$$

For a positive integer  $t$ , we define the DM-stack<sup>1</sup>  $\mathcal{Z}(t)$  by

$$\mathcal{Z}(t)(S) = \{(A, \iota, \lambda; x) \mid (A, \iota, \lambda) \in \mathcal{M}(\mathbf{k}; n-1, 1)(S), x \in V'(A, E), h'(x, x) = t\} .$$

Then  $\mathcal{Z}(t)$  maps by a finite unramified morphism to  $\mathcal{M}(\mathbf{k}; n-1, 1)_{\mathbb{C}}$ , and its image is a divisor in the sense that, locally for the étale topology, it is defined by a non-zero equation.

The cycles  $\mathcal{Z}(t)$  also admit a complex uniformization. More precisely, under the assumption of the triviality of the class group of  $\mathbf{k}$ , we have

$$\mathcal{Z}(t)(\mathbb{C}) \simeq \left[ \Gamma \backslash \left( \coprod_{\substack{x \in L \\ h(x, x) = t}} \mathcal{D}_x \right) \right],$$

where  $\mathcal{D}_x$  is the set of lines in  $\mathcal{D}$  which are perpendicular to  $x$ .

Again, there is a  $*$ -variant of these definitions and a corresponding DM-stack  $\mathcal{Z}(t)^*$  above  $\mathcal{M}(\mathbf{k}; n-1, 1)^*$ .

## 5. CUBIC SURFACES

In this paper we consider four occult period mappings. We start with the case of cubic surfaces, following Allcock, Carlson, Toledo [2], comp. also [5]. As explained in the introduction, in these sources, the results are formulated in terms of arithmetic ball quotients; here we use the complex uniformization of the previous two sections to express these results in terms of moduli spaces of Picard type.

Let  $S \subset \mathbb{P}^3$  be a smooth cubic surface. Let  $V$  be a cyclic covering of degree 3 of  $\mathbb{P}^3$ , ramified along  $S$ . Explicitly, if  $S$  is defined by the homogeneous equation of degree 3 in 4 variables

$$F(X_0, \dots, X_3) = 0 ,$$

then  $V$  is defined by the homogeneous equation of degree 3 in 5 variables,

$$X_4^3 - F(X_0, \dots, X_3) = 0 .$$

Let  $\mathbf{k} = \mathbb{Q}(\omega)$ ,  $\omega = e^{2\pi i/3}$ . Then the obvious  $\mu_3$ -action on  $V$  determines an action of  $O_{\mathbf{k}} = \mathbb{Z}[\omega]$  on  $H^3(V, \mathbb{Z})$ . For the (alternating) cup product pairing  $\langle \ , \ \rangle$ ,

$$\langle \omega x, \omega y \rangle = \langle x, y \rangle ,$$

which implies that

$$\langle ax, y \rangle = \langle x, a^{\sigma} y \rangle, \quad \forall a \in O_{\mathbf{k}} .$$

Hence there is a unique  $O_{\mathbf{k}}$ -valued hermitian form  $h$  on  $H^3(V, \mathbb{Z})$  such that

$$\langle x, y \rangle = \mathrm{tr} \left( \frac{1}{\sqrt{\Delta}} h(x, y) \right), \tag{5.1}$$

---

<sup>1</sup>This notation differs from that in [19], in that here the special cycles are defined over  $\mathbb{C}$ , and are considered as lying over  $\mathcal{M}(\mathbf{k}; n-1, 1)_{\mathbb{C}}$ .

where the discriminant  $\Delta$  of  $\mathbf{k}$  is equal to  $-3$  in the case at hand. Explicitly,

$$h(x, y) = \frac{1}{2}(\langle \sqrt{\Delta}x, y \rangle + \langle x, y \rangle \sqrt{\Delta}). \quad (5.2)$$

Furthermore, an  $O_{\mathbf{k}}$ -lattice is self-dual wrt  $\langle \cdot, \cdot \rangle$  if and only if it is self-dual wrt  $h(\cdot, \cdot)$ .

**Fact:**  $H^3(V, \mathbb{Z})$  is a self-dual hermitian  $O_{\mathbf{k}}$ -module of signature  $(4, 1)$ .

As noted above, such a lattice is unique up to isomorphism.

Let

$$A = A(V) = H^3(V, \mathbb{Z}) \backslash H^3(V, \mathbb{C}) / H^{2,1}(V)$$

be the intermediate Jacobian of  $V$ . Then  $A$  is an abelian variety of dimension 5 which is principally polarized by the intersection form. Since the association  $V \mapsto (A(V), \lambda)$  is functorial, we obtain an action  $\iota$  of  $O_{\mathbf{k}}$  on  $A(V)$ .

**Theorem 5.1.** (i) The object  $(A, \iota, \lambda)$  lies in  $\mathcal{M}(\mathbf{k}; 4, 1)(\mathbb{C})$ .

(ii) This construction is compatible with families and defines a morphism of DM-stacks,

$$\varphi : \text{Cubics}_{2, \mathbb{C}}^{\circ} \rightarrow \mathcal{M}(\mathbf{k}; 4, 1)_{\mathbb{C}}.$$

Here  $\text{Cubics}_{2, \mathbb{C}}^{\circ}$  denotes the stack parametrizing smooth cubic surfaces up to projective equivalence,

$$\text{Cubics}_{2, \mathbb{C}}^{\circ} = [\mathbb{P}\text{Sym}^3(\mathbb{C}^4)^{\circ} / \text{PGL}_4]$$

[stack quotient in the orbifold sense].

(iii) The morphism  $\varphi$  is an open embedding. Its image is the complement of the image of the KM-cycle  $\mathcal{Z}(1)$  in  $\mathcal{M}(\mathbf{k}; 4, 1)_{\mathbb{C}}$ .

*Proof.* We only comment on the assertions in (ii) and (iii). In (ii), the compatibility with families is always true of Griffiths' intermediate jacobians (which however are abelian varieties only when the Hodge structure is of type  $(m+1, m) + (m, m+1)$ ). This constructs  $\varphi$  as a complex-analytic morphism. The algebraicity of  $\varphi$  then follows from Borel's theorem that any analytic family of abelian varieties over a  $\mathbb{C}$ -scheme is automatically algebraic [6]. That one obtains an isomorphism of DM-stacks between  $\text{Cubics}_{2, \mathbb{C}}^{\circ}$  and  $\mathcal{M}(\mathbf{k}; 4, 1)_{\mathbb{C}} \setminus \mathcal{Z}(1)$ , is proved in [2], Thm. 2. 20. The fact that the image is contained in the complement of  $\mathcal{Z}(1)$  is true because, by the Clemens-Griffiths theory, intermediate Jacobians of cubic threefolds are simple as polarized abelian varieties, whereas, over  $\mathcal{Z}(1)$  the polarized abelian varieties split off an elliptic curve. However, the fact that  $\mathcal{Z}(1)$  makes up the whole complement is surprising and results from the fact that the morphism  $\varphi$  extends to an isomorphism from a partial compactification  $\text{Cubics}_{2, \mathbb{C}}^{\text{s}}$  of  $\text{Cubics}_{2, \mathbb{C}}^{\circ}$  (obtained by adding *stable* cubics) to  $\mathcal{M}(\mathbf{k}; 4, 1)_{\mathbb{C}}$ , such that the complement of  $\text{Cubics}_{2, \mathbb{C}}^{\circ}$  in  $\text{Cubics}_{2, \mathbb{C}}^{\text{s}}$  is an irreducible divisor, cf. [5], Prop. 6.7, Prop. 8.2.  $\square$

## 6. CUBIC THREEFOLDS

Our next example concerns cubic threefolds, following Allcock, Carlson, Toledo [3] and Looijenga, Swierstra [20].

Let  $T \subset \mathbb{P}^4$  be a cubic threefold. Let  $V$  be the cyclic covering of degree 3 of  $\mathbb{P}^4$ , ramified in  $T$ . Then  $V$  is a cubic hypersurface in  $\mathbb{P}^5$  and we define the primitive cohomology as

$$L = H_0^4(V, \mathbb{Z}) = \{x \in H^4(V, \mathbb{Z}) \mid (x, \rho) = 0\}, \quad (6.1)$$

where  $\rho$  is the square of the hyperplane section class. Note that  $\text{rk}_{\mathbb{Z}} L = 22$ . Again, let  $\mathbf{k} = \mathbb{Q}(\omega)$ , with  $\omega = e^{2\pi i/3}$ , so that  $L$  becomes an  $O_{\mathbf{k}}$ -module. Now the cup-product  $(\ , \ )$  on  $H^4(V, \mathbb{Z})$  is a perfect *symmetric* pairing satisfying  $(ax, y) = (x, a^\sigma y)$  for  $a \in O_{\mathbf{k}}$ . It induces on  $L$  a symmetric bilinear form  $(\ , \ )$  of discriminant 3. We wish to define an *alternating* pairing  $\langle \ , \ \rangle$  on  $L$  satisfying  $\langle ax, y \rangle = \langle x, a^\sigma y \rangle$  for  $a \in O_{\mathbf{k}}$ . We do this by giving the associated  $O_{\mathbf{k}}$ -valued hermitian pairing  $h(\ , \ )$ , in the sense of (5.1) defined by

$$h(x, y) = \frac{3}{2}((x, y) + (x, \sqrt{\Delta}y) \frac{1}{\sqrt{\Delta}}). \quad (6.2)$$

Here the factor  $3/2$  is used instead of  $1/2$  to have better integrality properties. Set  $\pi = \sqrt{\Delta}$ .

**Fact:** For the pairing (6.2),  $L^\vee$  contains  $\pi^{-1}L$  with  $L^\vee/\pi^{-1}L \simeq \mathbb{Z}/3\mathbb{Z}$ .

For this result, see [3], Theorem 2.6 and its proof, as well as [20], the passage below (2.1).

Now consider the eigenspace decomposition of  $H_0^4(V, \mathbb{C})$  under  $\mathbf{k} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$ .

**Fact:** The Hodge structure of  $H_0^4(V, \mathbb{R})$  is of type

$$H_0^4(V, \mathbb{C}) = H^{3,1} \oplus H_0^{2,2} \oplus H^{1,3},$$

with  $\dim H^{3,1} = \dim H^{1,3} = 1$ . Furthermore, the only nontrivial eigenspaces of the generator  $\omega$  of  $\mu_3$  are

$$\begin{aligned} H_0^4(V, \mathbb{C})_\omega &= H^{3,1} \oplus (H_0^{2,2})_\omega, \text{ with } \dim(H_0^{2,2})_\omega = 10 \\ H_0^4(V, \mathbb{C})_{\overline{\omega}} &= (H_0^{2,2})_{\overline{\omega}} \oplus H^{1,3}, \text{ with } \dim(H_0^{2,2})_{\overline{\omega}} = 10, \end{aligned}$$

see [3], §2, resp. [20] §4.

Now set  $\Lambda = \pi L^\vee$ . Then we have the chain of inclusions of  $O_{\mathbf{k}}$ -lattices

$$\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda,$$

where the quotient  $\Lambda^\vee/\Lambda$  is isomorphic to  $(\mathbb{Z}/3\mathbb{Z})^{10}$ , and where  $\pi^{-1}\Lambda/\Lambda^\vee$  is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ . Let

$$A = \Lambda \backslash H_0^4(V, \mathbb{C})/H^-,$$

where

$$H^- = H^{3,1} \oplus (H_0^{2,2})_{\overline{\omega}}.$$

Note that the map  $\Lambda \rightarrow H_0^4(V, \mathbb{C})/H^-$  is an  $O_{\mathbf{k}}$ -linear injection, hence  $A$  is a complex torus. In fact, the hermitian form  $h$  and its associated alternating form  $\langle \ , \ \rangle$  define a polarization  $\lambda$  on  $A$ . Hence  $A$  is an abelian variety of dimension 11, with an action of  $O_{\mathbf{k}}$  and a polarization of degree  $3^{10}$ . In fact, we obtain in this way an object  $(A, \iota, \lambda)$  of  $\mathcal{M}(\mathbf{k}; 10, 1)^*(\mathbb{C})$  (see section 2 for the definition of the  $*$ -variants of our moduli stacks).

**Theorem 6.1.** (i) The construction which associates to a smooth cubic  $T$  in  $\mathbb{P}^4$  the object  $(A, \iota, \lambda)$  of  $\mathcal{M}(\mathbf{k}; 10, 1)^*(\mathbb{C})$  is compatible with families and defines a morphism of DM-stacks,

$$\varphi : \text{Cubics}_{3, \mathbb{C}}^\circ \rightarrow \mathcal{M}(\mathbf{k}; 10, 1)_{\mathbb{C}}^*.$$

(ii) The morphism  $\varphi$  is an open embedding. Its image is the complement of the image of the KM-cycle  $Z(3)^*$  in  $\mathcal{M}(\mathbf{k}; 10, 1)_{\mathbb{C}}^*$ .

*Proof.* Again, we only give a few remarks. The compatibility with families is due to the fact that the eigenspaces for the  $\mu_3$ -action and the Hodge filtration both vary in a holomorphic way. The point (ii) is [3], Thm. 1.1, resp. [20], Thm. 3.1, except that the stack aspect is neglected in these sources. However, the fact that we have an open immersion of DM-stacks follows easily from the fact that the automorphism group of a cubic fourfold maps bijectively to the automorphism group of  $H^4$  (with its additional structure). Indeed, this fact implies that any element in the stabilizer group of a point in  $\text{Im}(\varphi)$  acts on the corresponding cubic fourfold, compatibly with the action of  $\mu_3$  on it. Hence it induces an automorphism of the ramification locus, which is identified with the cubic threefold corresponding to the preimage of the point under the period morphism.<sup>2</sup>  $\square$

**Remark 6.2.** The construction of the rational Hodge structure  $H^1(A, \mathbb{Q})$  from  $H_0^4(V, \mathbb{Q})$  is a very special case of a general construction due to van Geemen [11]. More precisely, it arises (up to Tate twist) as the *inverse half-twist* in the sense of loc. cit. of the Hodge structure  $H_0^4(V, \mathbb{Q})$  with complex multiplication by  $\mathbf{k}$ . The *half twist* construction attaches to a rational Hodge structure  $V$  of weight  $w$  with complex multiplication by a CM-field  $\mathbf{k}$  a rational Hodge structure of weight  $w + 1$ . More precisely, if  $\Sigma$  is a fixed half system of complex embeddings of  $\mathbf{k}$ , then van Geemen defines a new Hodge structure on  $V$  by setting

$$V_{\text{new}}^{r,s} = V_{\Sigma}^{r-1,s} \oplus V_{\bar{\Sigma}}^{r,s-1},$$

where  $V_{\Sigma}$ , resp.  $V_{\bar{\Sigma}}$  denotes the sum of the eigenspaces for the  $\mathbf{k}$ -action corresponding to the complex embeddings in  $\Sigma$ , resp. in  $\bar{\Sigma}$ .

## 7. CURVES OF GENUS THREE

Our third example concerns the moduli space of curves of genus 3 following Kondo [13].

Let  $C$  be a non-hyperelliptic smooth projective curve of genus 3. The canonical system embeds  $C$  as a quartic curve in  $\mathbb{P}^2$ . Let  $X(C)$  be the  $\mu_4$ -covering of  $\mathbb{P}^2$  ramified in  $C$ . Then the quartic  $X(C) \subset \mathbb{P}^3$  is a  $K3$ -surface with an automorphism  $\tau$  of order 4 and hence an action of  $\mu_4$ . Let

$$L = \{x \in H^2(X(C), \mathbb{Z}) \mid \tau^2(x) = -x\}.$$

Let  $\mathbf{k} = \mathbb{Q}(i)$  be the Gaussian field.

**Fact:**  $L$  is a free  $\mathbb{Z}$ -module of rank 14. The restriction  $(\ , \ )$  of the symmetric cup product pairing to  $L$  has discriminant  $2^8$ ; more precisely, for the dual lattice  $L^*$  for the symmetric pairing,

$$L^*/L \cong (\mathbb{Z}/2)^8,$$

see [13], top of p. 222.

Now consider the eigenspace decomposition of  $L_{\mathbb{C}} = L \otimes \mathbb{C}$  under  $\mathbf{k} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$ , where  $i \otimes 1$  acts via  $\tau$ .

**Fact:** The induced Hodge structure on  $L_{\mathbb{C}}$  is of type

$$L_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

with  $\dim H^{2,0} = \dim H^{0,2} = 1$ . Furthermore the only nontrivial eigenspaces of  $\tau$  are

$$\begin{aligned} (L_{\mathbb{C}})_i &= H^{2,0} \oplus (H^{1,1})_i, \text{ with } \dim(H^{1,1})_i = 6 \\ (L_{\mathbb{C}})_{-i} &= (H^{1,1})_{-i} \oplus H^{0,2}, \text{ with } \dim(H^{1,1})_{-i} = 6. \end{aligned}$$

<sup>2</sup> This argument is due to Looijenga.

We define an  $O_k$ -valued hermitian pairing  $h$  on  $L_{\mathbb{Q}}$  by setting

$$h(x, y) = (x, y) + (x, \tau y) i . \quad (7.1)$$

Then it is easy to see that the dual lattice  $L^{\vee}$  of  $L$  for the hermitian form  $h$  is the same as the dual lattice  $L^*$  for the symmetric form.

Now set  $\Lambda = \pi L^{\vee}$ , where  $\pi = 1 + i$ . Then we obtain a chain of inclusions of  $O_k$ -lattices

$$\Lambda \subset \Lambda^{\vee} \subset \pi^{-1} \Lambda ,$$

where the quotient  $\Lambda^{\vee}/\Lambda$  is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^6$ , and where  $\pi^{-1}\Lambda/\Lambda^{\vee}$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .

Let

$$A = \Lambda \backslash L_{\mathbb{C}} / H^{-} ,$$

where

$$H^{-} = H^{2,0} \oplus (H^{1,1})_{-i} .$$

Note that the map  $\Lambda \rightarrow L_{\mathbb{C}}/H^{-}$  is a  $O_k$ -linear injection, hence  $A$  is a complex torus. In fact, the hermitian form  $h$  and its associated alternating form  $\langle \cdot, \cdot \rangle$  define a polarization  $\lambda$  on  $A$ . Hence  $A$  is an abelian variety of dimension 7, with an action of  $O_k$  and a polarization of degree  $2^6$ . In fact, we obtain in this way an object  $(A, \iota, \lambda)$  of  $\mathcal{M}(k; 6, 1)^*(\mathbb{C})$ . Now [13], Thm. 2.5 implies the following theorem.

**Theorem 7.1.** *(i) The construction which associates to a non-hyperelliptic curve of genus 3 the object  $(A, \iota, \lambda)$  of  $\mathcal{M}(k; 6, 1)^*(\mathbb{C})$  is compatible with families and defines a morphism of DM-stacks,*

$$\varphi : \mathcal{N}_{3, \mathbb{C}}^{\circ} \rightarrow \mathcal{M}(k; 6, 1)_{\mathbb{C}}^* .$$

Here  $\mathcal{N}_{3, \mathbb{C}}^{\circ}$  denotes the stack of smooth non-hyperelliptic curves of genus 3, i.e., of smooth non-hyperelliptic quartics in  $\mathbb{P}^2$  up to projective equivalence.

*(ii) The morphism  $\varphi$  is an open embedding. Its image is the complement of the image of the KM-cycle  $\mathcal{Z}(2)^*$  in  $\mathcal{M}(k; 6, 1)_{\mathbb{C}}^*$ .*  $\square$

## 8. CURVES OF GENUS FOUR

Our final example concerns the moduli space of curves of genus four and is also due to Kondo [14].

Let  $C$  be a non-hyperelliptic curve of genus 4. The canonical system embeds  $C$  into  $\mathbb{P}^3$ . More precisely,  $C$  is the intersection of a smooth cubic surface  $S$  and a quartic  $Q$  which is either smooth or a quadratic cone. Furthermore,  $Q$  is uniquely determined by  $C$ . Let  $X$  be a cyclic cover of degree 3 over  $Q$  branched along  $C$  (in case  $Q$  is singular, we take the minimal resolution of the singularities, cf. loc.cit.). Then  $X$  is a  $K3$ -surface with an action of  $\mu_3$ . Let

$$L = (H^2(X, \mathbb{Z})^{\mu_3})^{\perp}$$

be the orthogonal complement of the invariants of this action in  $H^2(X, \mathbb{Z})$ , equipped with the symmetric form  $(\cdot, \cdot)$  obtained by restriction.

**Fact:**  $L$  is a free  $\mathbb{Z}$ -module of rank 20, with dual  $L^*$  for the symmetric form satisfying

$$L^*/L \simeq (\mathbb{Z}/3\mathbb{Z})^2 ,$$



cf. [14], top of p. 386.

For  $\mathbf{k} = \mathbb{Q}(\omega)$ ,  $\omega = e^{2\pi i/3}$ , we again define an alternating form  $\langle \cdot, \cdot \rangle$  through its associated  $O_{\mathbf{k}}$ -valued hermitian form  $h$ . Using the action of  $O_{\mathbf{k}}$  on  $L$ , we set

$$h(x, y) = \frac{3}{2} \left( (x, y) + (x, \sqrt{\Delta}y) \frac{1}{\sqrt{\Delta}} \right). \quad (8.1)$$

Set  $\pi = \sqrt{\Delta}$ .

**Fact:** For the hermitian pairing (8.1),  $L^\vee$  is an over-lattice of  $\pi^{-1}L$  with  $L^\vee / \pi^{-1}L \simeq (\mathbb{Z}/3\mathbb{Z})^2$ .

Now consider the eigenspace decomposition of  $L \otimes \mathbb{C}$  under  $\mathbf{k} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$ .

**Fact:** The induced Hodge structure on  $L_{\mathbb{C}}$  is of type

$$L_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

with  $\dim H^{2,0} = \dim H^{0,2} = 1$ . Furthermore the only nontrivial eigenspaces of  $\omega$  are

$$\begin{aligned} (L_{\mathbb{C}})_{\omega} &= H^{2,0} \oplus (H^{1,1})_{\omega}, \text{ with } \dim(H^{1,1})_{\omega} = 9 \\ (L_{\mathbb{C}})_{\bar{\omega}} &= (H^{1,1})_{\bar{\omega}} \oplus H^{0,2}, \text{ with } \dim(H^{1,1})_{\bar{\omega}} = 9. \end{aligned}$$

Now set  $\Lambda = \pi L^\vee$ . Then we have the chain of inclusions of  $O_{\mathbf{k}}$ -lattices

$$\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda,$$

where the quotient  $\Lambda^\vee / \Lambda$  is isomorphic to  $(\mathbb{Z}/3\mathbb{Z})^8$ , and where  $\pi^{-1}\Lambda / \Lambda^\vee$  is isomorphic to  $(\mathbb{Z}/3\mathbb{Z})^2$ .

Let

$$A = \Lambda \backslash L_{\mathbb{C}} / H^-,$$

where

$$H^- = H^{2,0} \oplus (H^{1,1})_{\bar{\omega}}.$$

Then the map  $\Lambda \rightarrow L_{\mathbb{C}} / H^-$  is a  $O_{\mathbf{k}}$ -linear injection, hence  $A$  is a complex torus. In fact, the hermitian form  $h$  and its associated alternating form  $\langle \cdot, \cdot \rangle$  define a polarization  $\lambda$  on  $A$ . Hence  $A$  is an abelian variety of dimension 10, with an action of  $O_{\mathbf{k}}$  and a polarization of degree  $3^8$ . In fact, we obtain in this way an object  $(A, \iota, \lambda)$  of  $\mathcal{M}(\mathbf{k}; 9, 1)^*(\mathbb{C})$ ,

**Theorem 8.1.** (i) The construction which associates to a non-hyperelliptic curve of genus 4 the object  $(A, \iota, \lambda)$  of  $\mathcal{M}(\mathbf{k}; 9, 1)^*(\mathbb{C})$  is compatible with families and defines a morphism of DM-stacks,

$$\varphi : \mathcal{N}_{4, \mathbb{C}}^\circ \rightarrow \mathcal{M}(\mathbf{k}; 9, 1)_{\mathbb{C}}^*.$$

Here  $\mathcal{N}_{4, \mathbb{C}}^\circ$  denotes the stack of smooth non-hyperelliptic curves of genus 4.

(ii) The morphism  $\varphi$  is an open embedding. Its image is the complement of the image of the KM-cycle  $Z(2)^*$  in  $\mathcal{M}(\mathbf{k}; 9, 1)_{\mathbb{C}}^*$ .  $\square$

## 9. DESCENT

In all four cases discussed above, we obtain morphisms over  $\mathbb{C}$  between DM-stacks defined over  $\mathbf{k}$ . These morphisms are constructed using transcendental methods. In this section we will show that these morphisms are in fact defined over  $\mathbf{k}$ . The argument is modelled on Deligne's solution of the analogous problem for complete intersections of Hodge level one [7], where he shows that the corresponding family of intermediate jacobians is an abelian scheme over the moduli scheme over  $\mathbb{Q}$  of complete intersections of given multi-degree.

In our discussion below, to simplify notations, we will deal with the case of cubic threefolds, as explained in section 6; the other cases are completely analogous. Below we will shorten the notation  $Cubics_3^\circ$  to  $\mathcal{C}$ , and consider this as a DM-stack over  $\text{Spec } \mathbf{k}$ . Let  $v : V \rightarrow \mathcal{C}$  be the universal family of cubic threefolds, and let  $a : A \rightarrow \mathcal{C}_{\mathbb{C}}$  be the polarized family of abelian varieties constructed from  $V$  in section 6. Hence  $A$  is the pullback of the universal abelian scheme over  $\mathcal{M}(\mathbf{k}; 10, 1)_{\mathbb{C}}^*$  under the open immersion  $\varphi : \mathcal{C}_{\mathbb{C}} \rightarrow \mathcal{M}(\mathbf{k}; 10, 1)_{\mathbb{C}}^*$ .

**Lemma 9.1.** *Let  $b : B \rightarrow \mathcal{C}_{\mathbb{C}}$  be a polarized abelian scheme with  $O_{\mathbf{k}}$ -action, which is the pullback under a morphism  $\psi : \mathcal{C}_{\mathbb{C}} \rightarrow \mathcal{M}(\mathbf{k}; 10, 1)_{\mathbb{C}}^*$  of the universal abelian scheme, and such that there exists  $\ell$  and an  $O_{\mathbf{k}}$ -linear isomorphism of lisse  $\ell$ -adic sheaves on  $\mathcal{C}_{\mathbb{C}}$ ,*

$$\alpha_{\ell} : R^1 a_* \mathbb{Z}_{\ell} \simeq R^1 b_* \mathbb{Z}_{\ell}$$

*compatible with the Riemann forms on source and target. Then there exists a unique isomorphism  $\alpha : A \rightarrow B$  that induces  $\alpha_{\ell}$ . This isomorphism is compatible with polarizations.*

To prove this, we are going to use the following lemma. In it, we denote by  $\Lambda$  the hermitian  $O_{\mathbf{k}}$ -module  $H^1(A_s, \mathbb{Z})$ , for  $s \in \mathcal{C}_{\mathbb{C}}$  a fixed base point. Recall from section 6 that there is a chain of inclusions  $\Lambda \subset \Lambda^{\vee} \subset \pi^{-1}\Lambda$ , where  $\pi = \sqrt{-3}$  is a generator of the unique prime ideal of  $O_{\mathbf{k}}$  dividing 3.

**Lemma 9.2.** *Let  $s \in \mathcal{C}_{\mathbb{C}}$  be the chosen base point.*

- (i) *The monodromy representation  $\rho_A : \pi_1(\mathcal{C}_{\mathbb{C}}, s) \rightarrow \text{GL}_{\mathbf{k}}(\Lambda \otimes_{O_{\mathbf{k}}} \mathbf{k})$  is absolutely irreducible.*
- (ii) *For every prime ideal  $\mathfrak{p}$  prime to 3, the monodromy representation  $\pi_1(\mathcal{C}_{\mathbb{C}}, s) \rightarrow \text{GL}_{\kappa(\mathfrak{p})}(\Lambda/\mathfrak{p}\Lambda)$  is absolutely irreducible.*
- (iii) *For the unique prime ideal  $\mathfrak{p} = (\pi)$  lying over 3, the monodromy representation  $\pi_1(\mathcal{C}_{\mathbb{C}}, s) \rightarrow \text{GL}_{\kappa(\mathfrak{p})}(\Lambda/\mathfrak{p}\Lambda)$  is not absolutely irreducible, but there is a unique non-trivial stable subspace, namely, the 10-dimensional image of  $\pi\Lambda^{\vee}$  in  $\Lambda/\pi\Lambda$ .*

*Proof.* The monodromy representations in question are induced by the composition of homomorphisms

$$\pi_1(\mathcal{C}_{\mathbb{C}}, s) \longrightarrow \pi_1(\mathcal{M}(\mathbf{k}; 10, 1)_{\mathbb{C}}^*, \varphi(s)) \longrightarrow \text{GL}_{O_{\mathbf{k}}}(H^1(A_s, \mathbb{Z})). \quad (9.1)$$

Here by Theorem 6.1, and using complex uniformization (cf. section 3), the first homomorphism is induced by the inclusion of connected spaces

$$\iota : \mathcal{D} \setminus \left( \bigcup_{\substack{x \in L \\ h(x, x) = 3}} \mathcal{D}_x \right) \hookrightarrow \mathcal{D},$$

followed by quotienting out by the free action of  $\Gamma^*$ . Since  $\mathcal{D}$  is simply-connected, it follows that  $\pi_1(\mathcal{M}(\mathbf{k}; 10, 1)_{\mathbb{C}}^*, \varphi(s)) = \Gamma^*$  and that the first homomorphism in (9.1) is surjective. Now,

$\Gamma^*$  can be identified with the group of unitary automorphisms of the *parahoric* lattice  $\Lambda$ , and it is elementary that the representations of  $\Gamma^*$  on  $\Lambda \otimes_{O_{\mathbf{k}}} \mathbf{k}$  and on  $\Lambda/\mathfrak{p}\Lambda$  for  $\mathfrak{p}$  prime to 3 are absolutely irreducible (the latter since  $\Lambda^\vee \otimes \mathbb{Z}_\ell = \Lambda \otimes \mathbb{Z}_\ell$  for  $\ell \neq 3$ ). The statement (iii) is proved in the same way.  $\square$

*Proof.* (of Lemma 9.1) Let us compare the monodromy representations,

$$\begin{aligned} \rho_A : \pi_1(\mathcal{C}_{\mathbb{C}}, s) &\rightarrow \mathrm{GL}_{O_{\mathbf{k}}}(H^1(A_s, \mathbb{Z})) \\ \rho_B : \pi_1(\mathcal{C}_{\mathbb{C}}, s) &\rightarrow \mathrm{GL}_{O_{\mathbf{k}}}(H^1(B_s, \mathbb{Z})). \end{aligned} \quad (9.2)$$

By hypothesis, these representations are isomorphic after tensoring with  $\mathbb{Z}_\ell$ . Hence, they are also isomorphic after tensoring with  $\mathbf{k}$ . Hence there exists a  $\pi_1(\mathcal{C}_{\mathbb{C}}, s)$ -equivariant  $\mathbf{k}$ -linear isomorphism

$$\beta : H^1(A_s, \mathbb{Q}) \simeq H^1(B_s, \mathbb{Q}).$$

By the irreducibility of the representation of  $\pi_1(\mathcal{C}_{\mathbb{C}}, s)$  in  $H^1(A_s, \mathbb{Q})$ ,  $\beta$  is unique up to a scalar in  $\mathbf{k}^\times$ . Let us compare the  $O_{\mathbf{k}}$ -lattices  $\beta^{-1}(H^1(B_s, \mathbb{Z}))$  and  $H^1(A_s, \mathbb{Z})$ . Since we are assuming that  $O_{\mathbf{k}}$  is a PID, after replacing  $\beta$  by a multiple  $\beta_{\mathcal{O}} = c\beta$ , we may assume that  $L_B = \beta_{\mathcal{O}}^{-1}(H^1(B_s, \mathbb{Z}))$  is a primitive  $O_{\mathbf{k}}$ -sublattice in  $\Lambda = H^1(A_s, \mathbb{Z})$ . Let  $\mathfrak{p}$  be a prime ideal in  $O_{\mathbf{k}}$ , and let us consider the image of  $L_B$  in  $\Lambda/\mathfrak{p}\Lambda$ . Since  $L_B$  is primitive in  $\Lambda$ , this image is non-zero. If  $\mathfrak{p}$  is prime to 3, the irreducibility statement in (ii) of Lemma 9.2 implies that this image is everything, and hence  $L_B \otimes O_{\mathbf{k}, \mathfrak{p}} = \Lambda \otimes O_{\mathbf{k}, \mathfrak{p}}$  in this case.

To handle the prime ideal  $\mathfrak{p} = (\pi)$  over 3, we use the polarizations. By the irreducibility statement in (i) of Lemma 9.2, the polarization forms on  $H^1(A_s, \mathbb{Q})$  and on  $H^1(B_s, \mathbb{Q})$  differ by a scalar in  $\mathbb{Q}^\times$  under the isomorphism  $\beta_{\mathcal{O}}$ . Now, by hypothesis on  $B$ , with respect to the polarization form on  $H^1(B_s, \mathbb{Q})$ , we have a chain of inclusions  $L_B \subset L_B^\vee \subset \pi^{-1}L_B$  with respective quotients of dimension 10 and 1 over  $\mathbb{F}_p$ , just as for  $\Lambda$ . Since the two polarization forms differ by a scalar, this excludes the possibility that the image of  $L_B$  in  $\Lambda/\pi\Lambda$  be non-trivial. It follows that  $L_B = \Lambda$ .

Furthermore, the isomorphism  $\beta_{\mathcal{O}}$  is unique up to a unit in  $O_{\mathbf{k}}^\times$ , and it is an isometry with respect to both polarization forms. Now, by [8], 4.4.11 and 4.4.12,  $\beta_{\mathcal{O}}$  is induced by an isomorphism of polarized abelian schemes. Finally,  $\beta_{\mathcal{O}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \alpha_\ell$  up to a unit, since these homomorphisms differ by a scalar and both preserve the Riemann forms.

The uniqueness of  $\alpha$  follows from Serre's Lemma.  $\square$

Now Lemma 9.1 implies that over any field extension  $k'$  of  $\mathbf{k}$  inside  $\mathbb{C}$ , there exists at most one polarized abelian variety  $b : B \rightarrow \mathcal{C}_{k'}$  obtained by pull-back from the universal abelian variety over  $\mathcal{M}(\mathbf{k}; 10, 1)^*$ , equipped with an  $O_{\mathbf{k}}$ -linear isomorphism of lisse  $\ell$ -adic sheaves over  $\mathcal{C}_{\mathbb{C}}$

$$R^1 a_* \mathbb{Z}_\ell \simeq R^1 b_{\mathbb{C}*} \mathbb{Z}_\ell,$$

preserving the Riemann forms. By the argument in [7], 2.2 this implies that, in fact,  $B$  exists (since it does for  $k' = \mathbb{C}$ ). Hence the morphism  $\varphi$  is defined over  $\mathbf{k}$ . Put otherwise, for any  $\mathbf{k}$ -automorphism  $\tau$  of  $\mathbb{C}$ , the conjugate embedding  $\varphi^\tau$ , which corresponds to the conjugate  $(A, \iota, l)^\tau$ , is equal to  $\varphi$ ; hence  $\varphi$  is defined over  $\mathbf{k}$ .

**Remark 9.3.** In the case of cubic surfaces, the original argument of Deligne [7] applies directly. Indeed, consider the following commutative diagram, in which the lower row is defined by associating to a cubic threefold its intermediate jacobian, a principally polarized abelian variety of

dimension 5,

$$\begin{array}{ccc} \text{Cubics}_{2\mathbb{C}}^{\circ} & \longrightarrow & \mathcal{M}(\mathbf{k}; 4, 1)_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \text{Cubics}_{3\mathbb{C}}^{\circ} & \longrightarrow & \mathcal{A}_{5\mathbb{C}} \end{array}.$$

By the Torelli theorem for cubic threefolds, this diagram is cartesian. The horizontal morphisms are open embeddings and the vertical morphisms are unramified. By [7], the lower horizontal morphism is defined over  $\mathbf{k}$  (even over  $\mathbb{Q}$ ). Hence also the upper horizontal morphism is defined over  $\mathbf{k}$ .

**Conjecture 9.4.** *In all four cases above, the morphisms  $\varphi$  can be extended over  $O_{\mathbf{k}}[\Delta^{-1}]$ .*

Since we circulated a first version of our paper, this has been proved by J. Achter [1] in the case of cubic surfaces.

## 10. CONCLUDING REMARKS

We end this paper with a few remarks.

**Remark 10.1.** In all four cases, the complement of  $\text{Im}(\varphi)$  is identified with a certain KM-divisor. In fact, for other KM-divisors, the intersection with  $\text{Im}(\varphi)$  sometimes has a geometric interpretation. For example, in the case of cubic surfaces, the intersection of  $\text{Im}(\varphi)$  with the image of the KM-divisor  $\mathcal{Z}(2)$  in  $\mathcal{M}(\mathbf{k}; 4, 1)_{\mathbb{C}}$  can be identified with the locus of cubic surfaces admitting Eckardt points, cf. [10], Thm. 8.10. Similarly, in the case of curves of genus 3, the intersection of  $\text{Im}(\varphi)$  with the image of  $\mathcal{Z}(t)^*$  in  $\mathcal{M}(\mathbf{k}; 6, 1)_{\mathbb{C}}$  can be identified with the locus of curves  $C$  where the K3-surface  $X(C)$  admits a “splitting curve” of a certain degree depending on  $t$ , cf. [4], Thm. 4.6.

**Remark 10.2.** In [9, 10, 22], occult period morphisms are often set in comparison with the Deligne-Mostow theory which establishes a relation between configuration spaces (e.g., of points in the projective line) and quotients of the complex unit ball by complex reflection groups, via monodromy groups of hypergeometric equations. This aspect of these examples has been suppressed entirely here. Also, it should be mentioned that there are other ways of constructing the period map for cubic surfaces, e.g., [9, 10].

**Remark 10.3.** Let us return to the section 3. There we had fixed an hermitian vector space  $(V, (\ , \ ))$  over  $\mathbf{k}$  of signature  $(n - 1, 1)$ . Let  $V_0$  be the underlying  $\mathbb{Q}$ -vector space, with the symmetric pairing defined by

$$s(x, y) = \text{tr}(h(x, y)).$$

Then  $s$  has signature  $(2(n - 1), 2)$ , and we obtain an embedding of  $U(V)$  into  $O(V_0)$ . This also induces an embedding of symmetric spaces,

$$\mathcal{D} \hookrightarrow \mathcal{D}_O, \tag{10.1}$$

where, as before,  $\mathcal{D}$  is the space of negative (complex) lines in  $(V_{\mathbb{R}}, \mathbb{I}_0)$ , and where  $\mathcal{D}_O$  is the space of oriented negative 2-planes in  $V_{\mathbb{R}}$ . The image of (10.1) is precisely the set of negative 2-planes that are stable by  $\mathbb{I}_0$ . In the cases of the Gauss field resp. the Eisenstein field, this invariance is equivalent to being stable under the action of  $\mu_4$ , resp.  $\mu_3$ . Hence in these two cases, the image of (10.1) can also be identified with the fixed point locus of  $\mu_4$  resp.  $\mu_3$  in  $\mathcal{D}_O$ .

**Remark 10.4.** By going through the tables in [23], §2, one sees that there is no further example of an occult period map of the type above which embeds the moduli stack of *hypersurfaces* of suitable degree and dimension into a Picard type moduli stack of abelian varieties. Note, however, that, in the case of curves of genus 4, the source of the hidden period morphism is a moduli stack of *complete intersections* of a certain multi-degree of dimension one, and there may be more examples of this type.

## REFERENCES

- [1] J. Achter, *Arithmetic Torelli maps for cubic surfaces and threefolds*, arXiv:1005.2131
- [2] D. Allcock, J. A. Carlson, D. Toledo, *The complex hyperbolic geometry of the moduli space of cubic surfaces*, J. Algebraic Geom., **11** (2002), no. 4, 659–724.
- [3] ———, *The moduli space of cubic threefolds as a ball quotient*, arXiv:math/0608287
- [4] M. Artebani, *Heegner divisors in the moduli space of genus three curves*, Trans. Amer. Math. Soc. **360** (2008), 1581–1599.
- [5] A. Beauville, *Moduli of cubic surfaces and Hodge theory (after Allcock, Carlson, Toledo)*, Géométries à courbure négative ou nulle, groupes discrets et rigidités, Séminaires et Congrès **18**, 446–467; SMF, 2008.
- [6] A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Diff. Geom., **6** (1972), 543–560.
- [7] P. Deligne, *Les intersections complètes de niveau de Hodge un*, Invent. math., **15** (1972), 237–250.
- [8] ———, *Théorie de Hodge II*, Publ. math. IHES, **40** (1971), 5–57.
- [9] I. V. Dolgachev, S. Kondo, *Moduli of K3 surfaces and complex ball quotients*, Arithmetic and geometry around hypergeometric functions, 43–100, Progr. Math., **260**, Birkhäuser, Basel, 2007.
- [10] I. V. Dolgachev, B. van Geemen, S. Kondo, *A complex ball uniformization of the moduli space of cubic surfaces via periods of K3 surfaces*, J. Reine Angew. Math., **588** (2005), 99–148.
- [11] B. van Geemen, *Half twists of Hodge structures of CM-type*, J. Math. Soc. Japan, **53** (2001), no. 4, 813–833.
- [12] B. van Geemen, E. Izadi, *Half twists and the cohomology of hypersurfaces*, Math. Z., **242** (2002), no. 2, 279–301.
- [13] S. Kondo, *A complex hyperbolic structure for the moduli space of curves of genus three*, J. Reine Angew. Math., **525** (2000), 219–232.
- [14] ———, *The moduli space of curves of genus 4 and Deligne-Mostow’s complex reflection groups*, Algebraic geometry 2000, Azumino (Hotaka), 383–400, Adv. Stud. Pure Math., **36**, Math. Soc. Japan, Tokyo, 2002.
- [15] S. Kudla, *Intersection numbers for quotients of the complex 2 ball and Hilbert modular forms*, Invent. math., **47** (1978), 189–208.
- [16] S. Kudla and J. Millson, *The theta correspondence and harmonic forms I*, Math. Annalen, **274** (1986), 353–378.
- [17] ———, *The theta correspondence and harmonic forms II*, Math. Annalen, **277** (1987), 267–314.
- [18] ———, *Intersection numbers for cycles in locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several variables*, Publ. math. IHES, **71** (1990), 121–172.
- [19] S. Kudla and M. Rapoport, *Special cycles on unitary Shimura varieties, II: global theory*, arXiv:0912.3758
- [20] E. Looijenga, R. Swierstra, *The period map for cubic threefolds*, Compos. Math., **143** (2007), no. 4, 1037–1049.
- [21] ———, *On period maps that are open embeddings*, J. Reine Angew. Math., **617** (2008), 169–192.
- [22] K. Matsumoto, T. Sasaki, M. Yoshida, *The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3, 6)*, Internat. J. Math., **3** (1992), no. 1, 164 pp.
- [23] M. Rapoport, *Complément à l’article de P. Deligne “La conjecture de Weil pour les surfaces K3”*, Invent. math., **15** (1972), 227–236.

Department of Mathematics  
University of Toronto  
40 St. George St., BA6290  
Toronto, ON M5S 2E4, Canada.  
email: skudla@math.toronto.edu

Mathematisches Institut der Universität Bonn  
Endenicher Allee 60  
53115 Bonn, Germany.  
email: rapoport@math.uni-bonn.de